16 990
LIBRARY

OFFICE OF NAVAL RESEARCH

Contract N7onr-35801

T. O. I.

NR-041-032

Technical Report No. 96

THE BURSTING SPEED OF A ROTATING PLASTIC DISC

by

H.J. Weiss and W. Prager

GRADUATE DIVISION OF APPLIED MATHEMATICS

BROWN UNIVERSITY

PROVIDENCE, R.I.

September, 1953

A11-96/18

A11-96

THE BURSTING SPEED OF A ROTATING PLASTIC DISC*

H. J. Weiss[1] and W. Prager[2]

Brown University

ABSTRACT

The paper presents an analysis of the stresses and strains in a fully plastic, rotating, annular disc that has initially uniform thickness and is made of a strain-hardening material. This analysis is based on Tresca's yield condition and the associated flow rule, and assumes that the elastic strains may be neglected in comparison to the finite plastic strains that are considered. The bursting speed of the disc is expressed in the form of a definite integral which involves the strain-hardening function of the material. In general, this integral will have to be evaluated numerically, but analytical evaluation is possible for certain strain-hardening functions. In particular, it is shown that for linear strain-hardening instability can occur only at the onset of plastic flow, whereas for logarithmic strain-hardening considerable plastic deformation of a stable character may occur before the process of deformation becomes unstable at the bursting speed.

---

*The results presented in this paper were obtained in the course of research sponsored by the Office of Naval Research under Contract N7onr-35801.

[1]Assistant Professor of Applied Mathematics.

[2]Professor of Applied Mechanics.

NOTATIONS

a_o, b_o	initial inner and outer radii of disc,
e_r, e_θ, e_z	instantaneous (true) rates of strain in the radial, circumferential, and axial directions,
h_o	initial uniform thickness of disc,
h	thickness of disc during plastic flow,
k, ℓ, m	constants in logarithmic stress-strain relation Eq. (16),
r_o	initial radial distance of a particle,
r	radial distance of a particle initially at r_o,
s_r, s_θ, s_z	nominal stress in the radial, circumferential, and axial directions,
s^*	critical nominal stress (initial yield stress),
u	radial displacement of a particle,
v	radial velocity of a particle,
$\alpha = b_o/a_o$,	
$\epsilon_r, \epsilon_\theta, \epsilon_z$	nominal strains in the radial, circumferential, and axial directions,
ϵ^*	critical nominal strain,
$\eta = u/a_o$,	
$\xi = r_o/a_o$,	
ρ	density of disc material,
$\sigma_r, \sigma_\theta, \sigma_z$	true stress in the radial, circumferential, and axial directions,
σ	twice the critical value of the true shearing stress,
ω	angular speed of the rotating disc.

(I) Introduction

The analysis of stresses and strains in elastic or partially plastic rotating discs has been discussed repeatedly [1 - 6]*. In these problems the strains are of the order of magnitude of elastic strains and may therefore be treated as infinitesimal. As a consequence of this, the equations of equilibrium may be satisfied for the undeformed rather than the deformed disc. Considerable simplification of the mathematical work results from this approximation. It would be unrealistic, however, to treat the strains as infinitesimal when the bursting speed of a strain-hardening plastic disc is to be determined. In fact, as the angular speed of such a disc is gradually increased, considerable plastic deformation of a stable character may occur before the process of deformation finally becomes unstable at the bursting speed.

Since the prediction of the bursting speed of a strain-hardening plastic disc requires the consideration of finite plastic strains, the analyst who has to forego the simplification resulting from infinitesimal strains will look for other possibilities of reducing the mathematical work. Thus, Zaid [7] uses a deformation theory of plasticity rather than a flow theory, in spite of his realization of the questionable value of this type of theory [8]. The alternative analysis offered in the present paper has recourse to other simplifying assumptions. Firstly, it is assumed that the elastic strains are negligible in comparison with the plastic strains.

---

*Numbers in square brackets refer to the bibliography at the end of the paper.

Secondly, it is assumed that the material of the disc obeys Tresca's yield condition [9] and the associated flow rule [10].

(II) TRESCA'S YIELD CONDITION AND FLOW RULE

In a rotationally symmetric stress field the principal stresses are the hoop stress σ_θ, the radial stress σ_r, and the axial stress σ_z. The principal shearing stresses therefore are $\frac{1}{2}|\sigma_\theta - \sigma_z|$, $\frac{1}{2}|\sigma_r - \sigma_z|$, and $\frac{1}{2}|\sigma_\theta - \sigma_r|$. According to Tresca's yield condition, none of these principal shearing stresses can exceed a critical value that depends on the stage of strain-hardening of the material. Moreover, for plastic flow to occur, at least one of the principal shearing stresses must have the critical value.

If only one of the principal shearing stresses has the critical value, the flow rule associated with Tresca's yield condition stipulates that the instantaneous strain rate corresponds to pure shear in the plane of maximum shearing stress. The sense of this shear deformation must be "appropriate", i.e., it must correspond to the sense of the maximum shearing stress.

If two principal shearing stresses attain the critical value, the flow rule admits any strain rate that can be considered as resulting from the superposition of appropriate states of pure shear in the two planes of critical shearing stress. It should be noted that in both cases the volume is preserved during plastic flow.

To express this yield condition and flow rule analytically, denote by $\sigma/2$ the critical value of the shearing stress for

the considered state of strain-hardening, and by e_θ, e_r, and e_z, the instantaneous rates of extension in the circumferential, radial, and axial directions. For the state of generalized plane stress occurring in a thin, rotating disc, the axial stress σ_z is zero. Moreover, for an annular disc of initially uniform thickness, it may be tentatively assumed that the hoop stress σ_θ and the radial stress σ_r are tensile and that σ_θ everywhere exceeds σ_r. Under these circumstances, Tresca's yield condition and the associated flow rule lead to the following relations:

$$\sigma_\theta = \sigma > \sigma_r > 0, \quad \sigma_z = 0, \tag{1}$$

$$e_r = 0, \quad e_\theta = - e_z \geq 0. \tag{2}$$

Since σ_θ and e_r are thus known explicitly, the number of unknown functions is reduced and the analysis is thereby simplified.

To complete the description of the assumed mechanical behavior of the disc material, it is necessary to formulate a law of strain-hardening. For simple tension, let the stress-strain diagram have the general shape of the line OAB in Fig. 1. The material remains rigid until the tensile stress s reaches the value s*; during the ensuing plastic flow, the stress increases with the strain ϵ according to

$$s = f(\epsilon). \tag{3}$$

The rate of strain-hardening ds/de has the initial value s*/ϵ* (see Fig. 1) and decreases monotonically with increasing strain.

In Eq. (3), s and ϵ should be interpreted as the conventional stress and strain computed with reference to the original

dimensions of the test specimen. On the other hand, the quantities σ_θ, σ_r, σ_z, σ, e_θ, e_r, e_z introduced above should be interpreted as the true stresses and strain rates; they are defined with respect to the dimensions in the considered state of deformation.

It remains to discuss the application of the strain-hardening law (3) to the states of stress and plastic flow occurring in a rotating disc. The state of plastic flow described by (2) is independent of the value of the intermediate principal stress σ_r. Moreover, this type of plastic flow is possible under Tresca's flow rule when the state of stress is simple tension in the circumferential direction. It is therefore reasonable to assume that Eq. (3) can be applied to the problem on hand, provided that s is interpreted as the conventional hoop stress s_θ and ε as the conventional hoop strain ε_θ.

(III) FULLY PLASTIC DISC

Consider a rotating annular disc of the uniform initial thickness h_o, the initial interior radius a_o, and the initial exterior radius b_o. From the elastic stress analysis it is known that, as the angular speed ω of the disc is gradually increased, the yield limit is first reached at the interior surface. At somewhat higher speeds there will be an inner plastic region surrounded by an elastic region. Any flow that may occur in the plastic region would have to satisfy (2). If $v = v(r)$ is the distribution of the radial velocity at a generic instant during this plastic flow, the radial strain rate is $e_r = \partial v/\partial r$. Since this must vanish according to (2), the radial velocity must be independent of r. Moreover,

since all elastic strains are neglected, the radial displacement u at the elastic-plastic interface is zero. This means that v and hence the circumferential strain rate $e_\theta = v/r$ must vanish throughout the plastic region. This region thus remains rigid until the elastic-plastic interface reaches the exterior surface of the disc. The angular speed ω_1 at which this occurs is given by (see Ref. 6, p. 103, Eq. 9.58)

$$\omega_1^2 = \frac{3s^*}{\rho} \frac{b_o - a_o}{b_o^3 - a_o^3}, \tag{4}$$

where ρ is the density of the disc material. The fully plastic stress distribution at the onset of plastic flow is readily analyzed (see Ref. 6, p. 104, Eq. 9.59 and Fig. 9.5). It is found that, at this instant, the radial stress nowhere reaches the yield stress regardless of the values of $b_o > a_o > 0$. Thus, (1) and (2) apply, initially at least, during the ensuing plastic flow.

Let r_o denote the initial radius of a particle that is found at the radius r when the speed is ω. If the radial displacement is denoted by u, then

$$r = r_o + u. \tag{5}$$

Here, r and u may be considered as functions of the independent variables r_o and ω^2. (It is convenient to use ω^2 rather than ω, since the sense of the plastic flow is independent of the sense of rotation or the sign of the angular speed.)

As has been shown above, the first Eq. (2) requires that all particles have the same radial velocity at any given instant.

This implies that the radial displacement depends only on ω^2 but not on r_o. Thus, the material bounded in the undeformed state by the coaxial cylinders of radii r_o and $r_o + dr_o$ undergoes the radial displacement u and is then bounded by cylinders of radii $r = r_o + u$ and $r + dr = r + dr_o$ in the deformed state. Simultaneously, the original thickness h_o decreases to h. Because of the incompressibility of the material, $h_o r_o dr_o = hrdr$ or, since $dr = dr_o$

$$h = h_o \frac{r_o}{r} = h_o \frac{r_o}{r_o + u} . \tag{6}$$

The true radial stress σ_r is transmitted across a section that is proportional to hr. Since $hr = h_o r_o$, by (6), the conventional and true radial stresses have the same value: $\sigma_r = s_r$. The true circumferential stress σ_θ is transmitted across a section that is proportional to hdr; the corresponding value in the undeformed state is $h_o dr_o = h_o dr$. The conventional circumferential stress is therefore given by

$$s_\theta = \sigma_\theta \frac{h}{h_o} . \tag{7}$$

This stress is related to the conventional circumferential strain

$$\varepsilon_\theta = \frac{u}{r_o} \tag{8}$$

by an equation of the form (3).

The equation of equilibrium in the deformed state is

$$-\frac{\partial(rh\sigma_r)}{\partial r} = h\sigma_\theta - h\rho\omega^2 r^2. \tag{9}$$

With reference to the undeformed state this equation may be written as follows:

$$\frac{\partial(r_o s_r)}{\partial r_o} = s_\theta - \rho\omega^2 r_o(r_o + u)$$

$$= f\left(\frac{u}{r_o}\right) - \rho\omega^2 r_o(r_o + u). \qquad (10)$$

Since s_r vanishes at the interior and exterior surfaces of the disc, i.e., for $r_o = a_o$ and $r_o = b_o$, the integral of the right-hand side of (10) between the limits a_o and b_o must vanish. With $\alpha = b_o/a_o$, $\eta = u/a_o$, and $\xi = r_o/a_o$, this condition yields

$$\frac{\rho a_o^2\omega^2}{6} = \frac{\int_1^\alpha f(\eta/\xi)d\xi}{2(\alpha^3 - 1) + 3\eta(\alpha^2 - 1)}. \qquad (11)$$

For a known strain-hardening function f and given initial dimensions, the right-hand side of (11) must be evaluated, analytically or numerically, for a set of values of $\eta = u/a_o$. Each such evaluation furnishes one point of the plot ω^2 vs. u. For $\eta = 0$, in particular, $f(0) = s^*$ and the integral in (11) equals $(\alpha - 1)s^*$. Equation (11) therefore yields the correct value (4) for the angular speed at which plastic flow begins.

As long as an increase in ω^2 is required to produce an increase in u the considered plastic flow is stable. The bursting speed ω_2 corresponds to the maximum of the curve ω^2 vs. u.

The preceding analysis is based on the assumption that the radial stress σ_r is smaller than the circumferential stress σ_θ. A formula for the maximum radial stress occurring at a given speed ω can be obtained by carrying out the differentiation on the left-hand side of (10) and equating $\partial s_r/\partial r_o$ to zero. Thus,

$$\max s_r = s_\theta - \rho\omega^2 r_o(r_o + u), \tag{12}$$

where r_o is now the initial value of the radius at which the maximum of s_r occurs. Equation (12) shows that even this maximum of s_r is smaller than the corresponding value of s_θ. Now, $\sigma_r = s_r$ and $\sigma_\theta > s_\theta$, by Eq. (7). The maximum of σ_r is therefore smaller than the corresponding value of σ_θ.

(IV) LINEAR STRAIN-HARDENING

As a rule, the integral in (11) must be evaluated numerically, but strain-hardening laws can be devised that make analytical evaluation possible. The simplest law of this kind is represented by the line OAC of Fig. 1. With

$$f(\varepsilon) = s^*(1 + \varepsilon/\varepsilon^*), \tag{13}$$

Eq. (11) yields

$$\frac{\rho a_o^2 \omega^2}{6s^*} = \frac{\alpha - 1 + (\eta/\varepsilon^*)\log\alpha}{2(\alpha^3 - 1) + 3\eta(\alpha^2 - 1)}. \tag{14}$$

Differentiation of (14) with respect to η shows that $d(\omega^2)/d\eta = 0$ when

$$\varepsilon^* = \frac{2(\alpha^2 + \alpha + 1)\log\alpha}{3(\alpha^2 - 1)}. \tag{15}$$

Equation (15) establishes the critical rate of linear strain-hardening. For a given value of $\alpha = b_o/a_o$, the strain-hardening parameter ε^* (see Fig. 1) must not exceed the value (15) if plastic deformation is to be stable. In Fig. 2, ε^* as given by (15) is plotted versus α. As $\alpha \to 1$, i.e., as the width of the

annulus tends towards zero, the critical value of ε^* tends towards 1. This result is familiar from Laszlo's paper [11].

The sign of $d(\omega^2)/d\eta$ as found from (14) does not change with increasing η. This means that for linear strain-hardening instability can occur only at the onset of plastic flow. A disc that starts to flow in a stable manner will continue to do so as the angular speed is increased.

(V) LOGARITHMIC STRAIN-HARDENING

Another strain-hardening law for which the integral in (11) can be evaluated analytically is given by

$$f(\varepsilon) = k \log (\ell + m\varepsilon), \tag{16}$$

where k, ℓ, and m are constants. These may be chosen so as to fit an experimental stress-strain curve. With the function (16), Eq. (11) yields

$$\frac{\rho a_o^2 \ell \omega^2}{6k} = \frac{(\alpha\ell + m\eta)\log(\alpha\ell + m\eta) - (\ell + m\eta)\log(\ell + m\eta) - \alpha\ell \log \alpha}{2(\alpha^3 - 1) + 3\eta(\alpha^2 - 1)}. \tag{17}$$

Figure 3 shows the actual stress-strain curve* for Al 24S-T4 in simple tension (full line) and the curve obtained from Eq. (16) with k = 9,780 psi, ℓ = 77, m = 8,060. Figure 4 shows the left-hand side of (17) vs. $\eta = u/a_o$ for $\alpha = b_o/a_o = 2$. The bursting speed ω_2 is obtained from the maximum point of this curve. As can be seen, the disc bursts for $\eta = \eta_2 = 0.25$. Since $\eta = u/a_o$, and

---

*The authors are indebted to Professor E. D'Appolonia, Carnegie Institute of Technology, Pittsburgh, Pa., for the use of this curve.

$\epsilon = u/r_o$, $\epsilon = \eta\ a_o/r_o$; and since the maximum nominal hoop strain occurs at $r_o = a_o$, the disc would burst at a hoop strain of 25%. From Fig. 3, it can be seen however, that this hoop strain is beyond the range of validity of the experimental stress-strain curve. A more realistic picture is obtained by integrating the right-hand side of (11) numerically, using the experimental stress-strain curve in Fig. 3 (full line) as $s = f(\epsilon)$. Figure 5 shows a plot of $\rho a_o^2\omega^2/6s^*$ (where s^* is as indicated in Fig. 3) vs. η, for $\alpha = 2$. Again, the bursting speed, ω_2, is obtained from the maximum point of this curve, which occurs at $\eta = \eta_2 = 0.14$, and thus at a maximum nominal hoop strain of 14%. This maximum strain occurs before necking takes place in the test specimen, and thus is in the range of validity of the experimental stress-strain curve. The results just obtained illustrate the fact that, for logarithmic strain-hardening as well as actual strain-hardening of a similar type, there is a considerable range of stable plastic deformation before bursting occurs.

REFERENCES

1. Timoshenko, S., and Goodier, J. N., Theory of Elasticity, McGraw-Hill Book Co., Inc., New York, 1951, pp. 69-73 and 352-354.

2. Nadai, A., and Donnell, L. H., Stress Distribution in Rotating Disks of Ductile Material After the Yield Point Has Been Reached, Transactions, ASME, APM-51-61, 1929.

3. Nadai, A., Theory of Flow and Fracture of Solids, McGraw-Hill Book Co., Inc., New York, 1950, pp. 482-489.

4. Lee Wu, M. H., General Plastic Behavior and Approximate Solutions of Rotating Disk in Strain-Hardening Range, NACA, Rep. 1021, May, 1951.

5. Zaid, M., Partially Plastic Discs, Journal of the Aeronautical Sciences, Vol. 19, No. 10, pp. 697-704, October, 1952.

6. Hoffman, O., and Sachs, G., Introduction to the Theory of Plasticity for Engineers, McGraw-Hill Book Co., Inc., New York, 1953, pp. 96-104.

7. Zaid, M., Fully Plastic Rotating Disc with Large Strains, Journal of the Aeronautical Sciences, Vol. 20, No. 6, pp. 369-377, June, 1953.

8. Prager, W., Theory of Plastic Flow Versus Theory of Plastic Deformation, Journal of Applied Physics, Vol. 19, No. 6, pp. 540-543, June, 1948.

9. Tresca, H., Mémoire sur l'Ecoulement des Corps Solides, Mémoires Presente's par Divers Savants, Vol. 18, pp. 733-799, 1868.

10. Prager, W., On the Use of Singular Yield Conditions and the Associated Flow Rules, to appear in the Journal of Applied Mechanics.

11. Laszlo, F., Geschleuderte Umdrehungskoerper im Gebiet bleibennder Deformation, Zeitschrift fuer angewandte Mathematik und Mechanik, Vol. 5, pp. 281-293, No. 4, August, 1929.

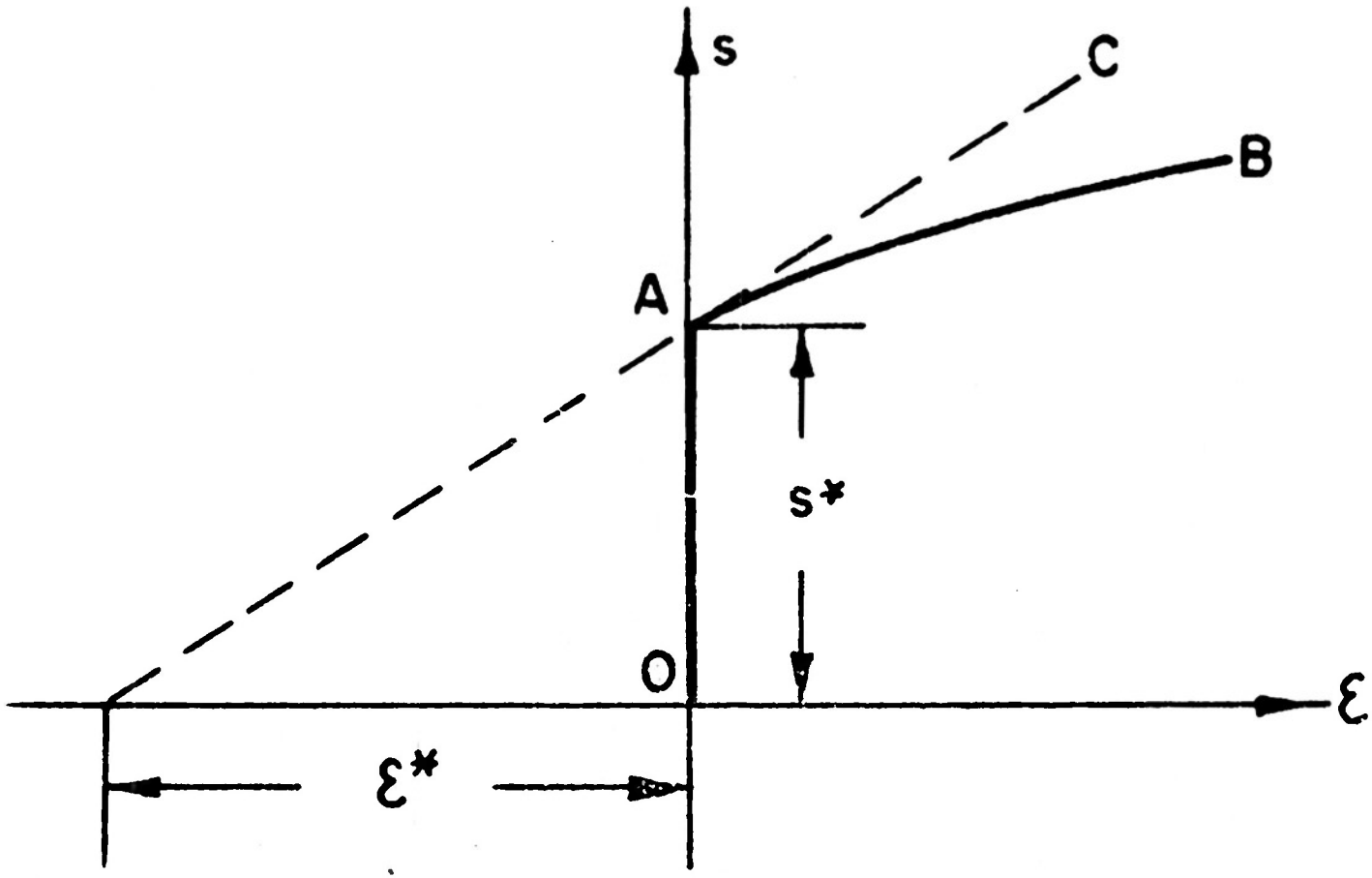

FIG. 1

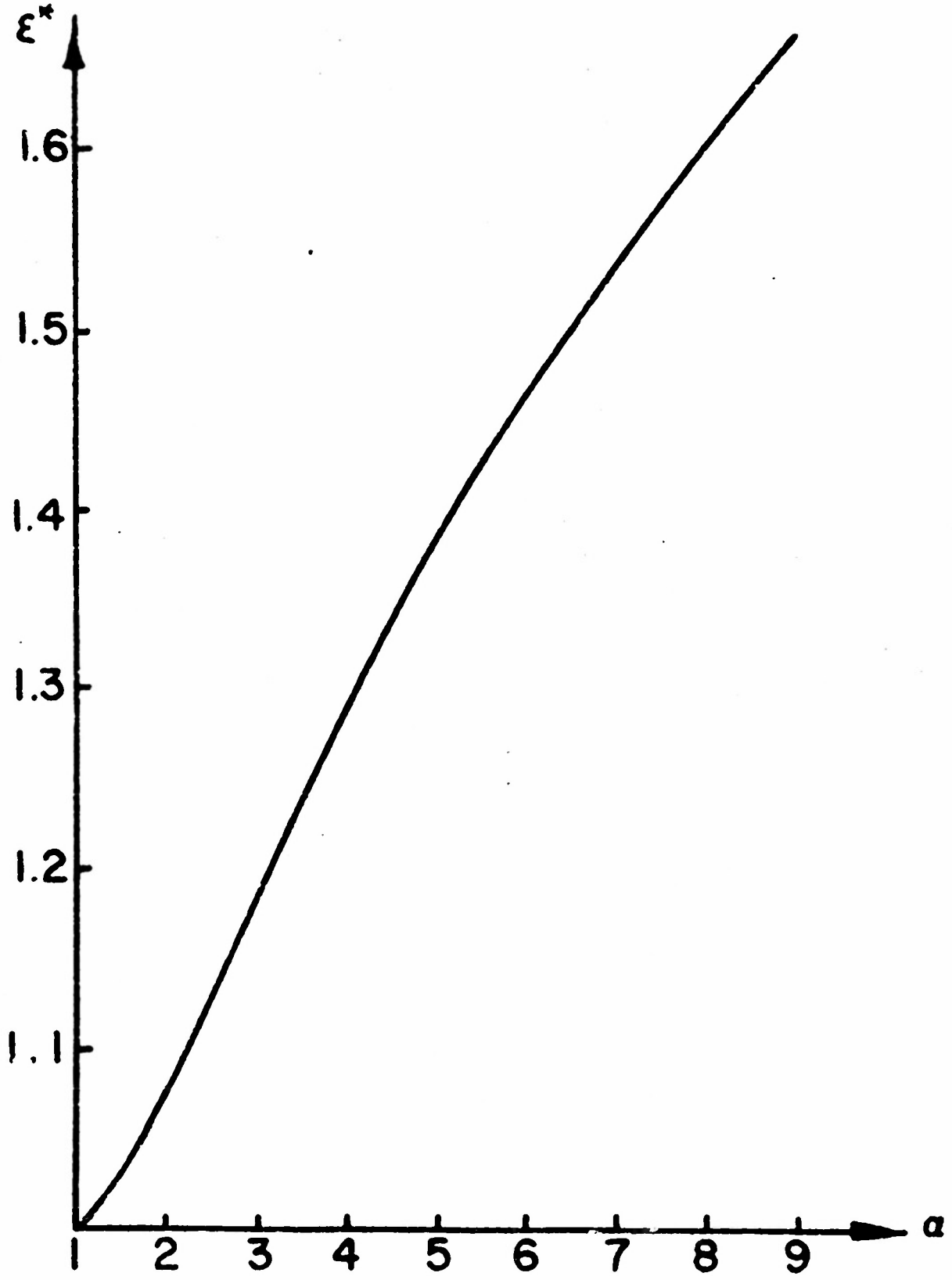

FIG. 2

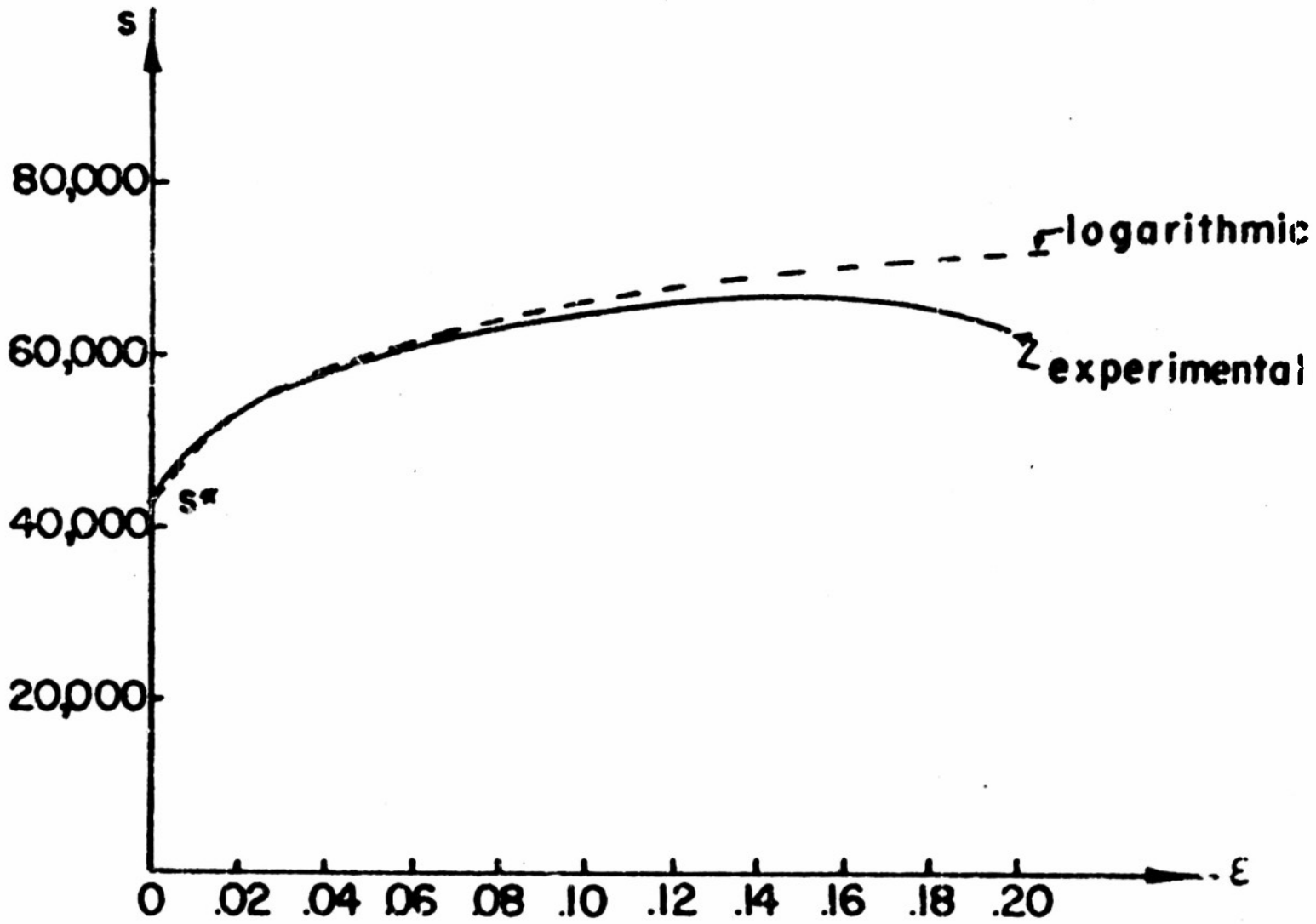
S
80,000
60,000
40,000
20,000
S^*
logarithmic
experimental
0
.02
.04
.06
.08
.10
.12
.14
.16
.18
.20
ε

FIG. 3

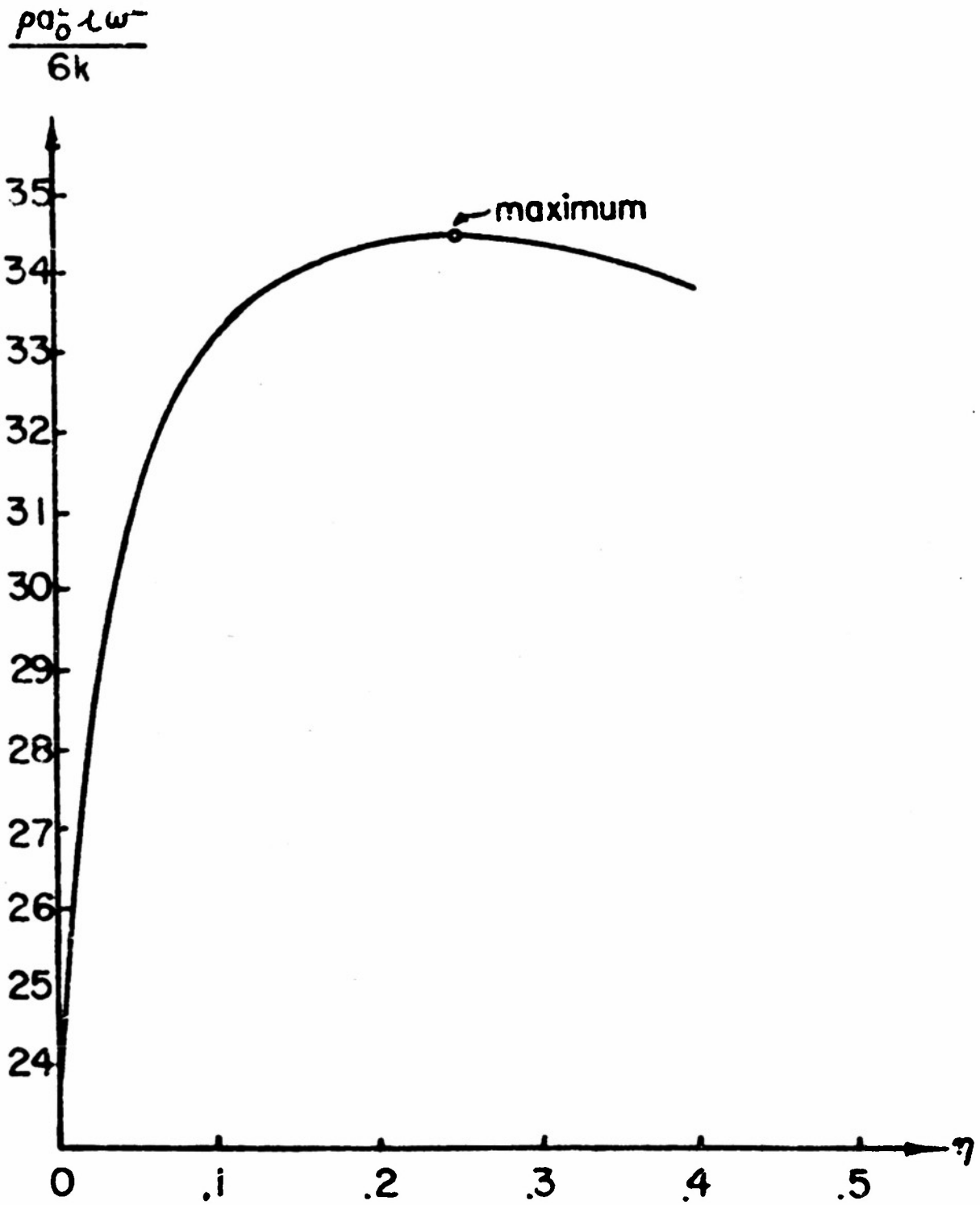
$\frac{\rho a_0^2 \omega^2}{6k}$
35
34
33
32
31
30
29
28
27
26
25
24
maximum
0
.1
.2
.3
.4
.5
η

FIG. 4

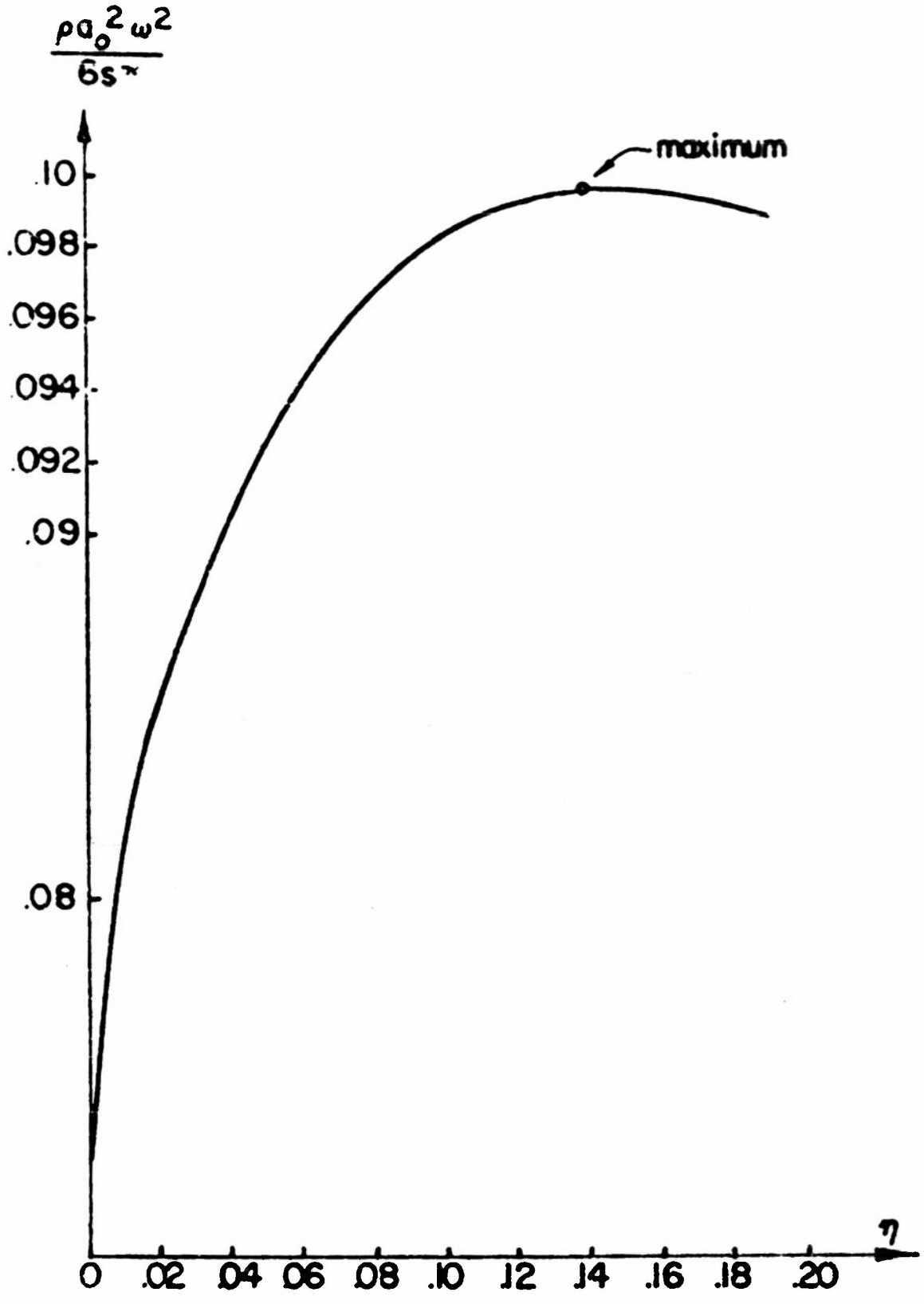
$\frac{\rho a_0^2 \omega^2}{6s^*}$
maximum
.10
.098
.096
.094
.092
.09
.08
η
0
.02
.04
.06
.08
.10
.12
.14
.16
.18
.20

FIG. 5